Journal of Applied Mechanics and Technical Physics, Vol. 40, No. 3, 1999

STABILITY OF A LAMINAR RIVULET LIQUID FLOW IN A CYLINDRICAL DUCT IN THE APPROXIMATION OF ONE-DIMENSIONAL WAVES

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The problem of the stability of a viscous laminar liquid flow with a liquid free surface in an inclined duct is theoretically considered. Since the dependence of the flow rate on the free-surface height is not monotonic (the highest flow rate in a cylindrical duct is observed at $H_* = 1.7R$), primary attention is given to the region $H > H_*$. It is proved that there is a region of instability: for an arbitrarily low Reynolds number, there is a free-surface level above which the flow becomes unstable against one-dimensional disturbances. When the height of the liquid layer is close to the vertical dimension of the duct, the one-dimensional disturbances propagate mainly upstream (for moderate Reynolds numbers). Hence it follows that there is no steady regime of liquid flow from a fully filled duct with an open end.

Introduction. The present paper deals with a theoretical study of the stability of a viscous laminar liquid flow with a free surface in an inclined duct. Previous numerical calculations [1, 2] of the parameters of such flows in a steady regime revealed a nonmonotonic behavior of the flow rate as a function of the free-surface height (Fig. 1). It was established that in a cylindrical duct, the flow rate attains a maximum for a layer height $H = 1.7R$, where R is the duct radius (curve 2, ratio of duct axes $1:1.0$). In ducts with an elliptic cross section extended in the horizontal direction, this maximum is more pronounced (curve 1, ratio of duct axes $1:1.3$). However, if the cross section is elliptic and extended along the vertical, there can be no maximum (curve 3, ratio of duct axes $1:0.7$). The critical ratio of duct axes is about $1:0.6$. The calculations were carried out using various boundary integral equation methods, and, in particular, the complex method of boundary elements [3].

This behavior of the flow rate can be qualitatively explained by the effect due to the duct walls. Indeed, if the liquid occupies the entire cross-sectional area, the walls exert a more intense decelerating effect on the flow than in the case of $H = 1.7R$ (a cylindrical duct). In the first case, the velocity of liquid particles is maximal at the center of the duct cross section, whereas in the second, the maximum is shifted toward the free surface, and, hence, the mean fluid velocity is higher in this case.

Various aspects of the stability of two-phase laminar ducted flows have been studied, e.g., in [4–6], but the problem of salient features of flows in which the height of the liquid layer exceeds the level corresponding to the maximum flow rate has not been posed. However, the nonmonotonic behavior of the flow rate suggests that this flow region can be unstable even at low Reynolds numbers. Indeed, we assume that in a certain segment of the duct, the free surface rises from a level $H = 1.8R$ to a level $H = 1.9R$ as a result of a fluctuation. The flow rate decreases locally compared to its initial value, which inevitably gives rise to a compression wave, "packing" of the liquid, and, ultimately, to full blocking of the duct in the case of a laminar flow. Thus, in a region where the height of the liquid layer exceeds the level that corresponds to the highest flow rate, any rise in the free-surface level leads to its further increase. At the same time, any decrease in the layer height (in the same region) results in a local increase in the flow rate, "drawing-off" of the liquid flowing above the level of the maximum flow rate, and, hence, further lowering of the free-surface level down to a height $H = 1.7R$.

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0021-8944/99/4003-0439 $22.00

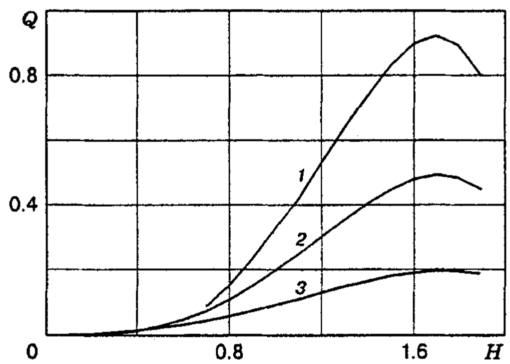

Fig. 1

This work is performed to verify this assumption. Since the formulated problem is rather complex, it is studied in the simple approximation of long, one-dimensional waves. To close the system of averaged dynamic equations, we use an approach which is an improved version of the method of [7] as applied to stability studies of liquid flows in the outer portion of a smooth, inclined cylinder.

1. Derivation of the Basic Equations. We consider a flow with a free surface in a cylindrical duct. The following notation is used: α is the slope of the duct to the horizontal, H is the maximum height of the free surface, $S(x,t)$ is the cross-sectional area of the duct occupied by the fluid, and L is the width of the free surface. The x, y, and z axes are directed downstream along the duct axis, along the vertical, and in the transverse direction, respectively, and the velocity components $\boldsymbol{v} = (u, v, w)$ correspond to these coordinates.

In the stability analysis, the following model is adopted as a first approximation:

1) The waves are assumed long ($kH \ll 1$, where k is the wavenumber);

2) wetting is ignored;

3) the waves are considered one-dimensional: $H(x,y,z,t) \equiv H(x,t)$ and $dS\,(x,t) \equiv L(H)\,dH$;

4) since primary attention is focussed on the region $H > 1.6R$, all parameters selected are those best suited to this section.

To obtain the governing dynamic equations, we use the following Navier–Stokes equation for the streamwise velocity and the continuity equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\Delta u + g\sin\alpha, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{1.1}$$

Here ν is the kinematic viscosity, ρ is the liquid density, and g is the free-fall acceleration.

The boundary conditions for the velocity components are the nonpenetration and attachment conditions at the duct wall, the absence of friction at the free surface ($\partial u/\partial n = 0$), and the kinematic condition at the free surface for the vertical velocity component:

$$v_H \equiv \frac{dH}{dt} = \frac{\partial H}{\partial t} + u_H\frac{\partial H}{\partial x}.$$

Averaging Eqs. (1.1) over the cross-sectional area with allowance for the boundary conditions, we obtain the system

$$\frac{\partial Q}{\partial t} + \frac{\partial Q_2}{\partial x} + \frac{1}{\rho}\iint\frac{\partial p}{\partial x}dS = \oint\nu\frac{\partial u}{\partial n}dl + Sg\sin\alpha, \quad \frac{\partial Q}{\partial x} + L\frac{\partial H}{\partial t} = 0. \tag{1.2}$$

Here Q and Q_2 denote the specific mass and momentum fluxes:

$$Q = \iint u\,dy\,dz, \qquad Q_2 = \iint u^2\,dy\,dz.$$

Next, the closure problem arises: it is required to express the friction force, the averaged pressure gradient, and the momentum flux in terms of S and Q. Following [7], to close the system, we use the hydrostatic

TABLE 1

H_0	Q_0	γ	β_1	β_2	β_3	f_1	ζ
0.10	0.0001	2.322	25.10	37.26	32.67	0.995	−2.96
0.20	0.0014	1.384	19.66	28.86	25.39	0.989	−0.17
0.30	0.0052	1.348	16.86	24.45	21.58	0.981	−0.07
0.40	0.0132	1.341	15.23	21.80	19.32	0.973	−0.06
0.50	0.0266	1.338	14.20	20.02	17.82	0.962	−0.06
0.60	0.0465	1.336	13.50	18.73	16.76	0.949	−0.07
0.70	0.0736	1.335	13.05	17.78	16.00	0.933	−0.09
0.80	0.1079	1.336	12.76	17.05	15.43	0.913	−0.12
0.90	0.1491	1.337	12.61	16.46	15.01	0.887	−0.15
1.00	0.1963	1.339	12.57	16.00	14.70	0.855	−0.19
1.10	0.2480	1.341	12.64	15.62	14.50	0.811	−0.24
1.20	0.3021	1.344	12.83	15.32	14.39	0.753	−0.29
1.30	0.3558	1.346	13.13	15.07	14.34	0.672	−0.36
1.40	0.4060	1.348	13.59	14.85	14.37	0.553	−0.43
1.50	0.4485	1.349	14.24	14.63	14.49	0.371	−0.47
1.60	0.4792	1.348	15.16	14.40	14.69	0.068	−0.39
1.70	0.4929	1.344	16.43	14.02	14.93	−0.498	0.24
1.80	0.4843	1.335	18.28	13.26	15.15	−1.798	4.10
1.90	0.4493	1.325	21.16	11.37	15.06	−6.467	45.70
1.98	0.4018	1.337	23.19	9.05	14.38	−55.281	3092.96

approximation (for pressure) and quasistationary approximation (for friction and momentum flux), which are applicable for rather long waves:

$$Q_2 = \gamma \frac{Q^2}{S}; \tag{1.3}$$

$$\oint \nu \frac{\partial u}{\partial n}\, dl = -\beta_1 \nu \frac{Q}{S}; \tag{1.4}$$

$$p = p_0 + \rho g \cos\alpha\,(H - R - y) - \sigma \frac{\partial^2 H}{\partial x^2}. \tag{1.5}$$

Here σ is the surface-friction coefficient and γ and β_1 are dimensionless constants. Expression (1.5) is obtained in the hydrostatic approximation from the Navier–Stokes equation for the vertical velocity component. The term describing the surface friction along the x axis is of the second order of smallness in this model, but it is retained in the formula to predict the behavior of the solution with increase in kH to values of the first order of smallness.

As follows from Table 1, hypothesis (1.4) is not quite adequate since β_1 can be considered constant only in the region $0.4R < H < 1.6R$, and, outside it, β_1 changes by 7–15% as the height changes by $0.1R$. Can the closure hypothesis be corrected so that this parameter, which characterizes the ratio of the friction force to the liquid flow rate, would exhibit a more conservative behavior?

It is difficult to derive an expression applicable over the entire range of heights from 0 to $2R$ since as the free-surface level rises, the key parameters behave differently. For example, the cross-sectional area increases monotonically, whereas L and Q take maximum values at a certain height, after which they begin to decrease, L varying from 0 (at $H = 0$ and $H = 2R$) to $2R$ (at $H = R$).

However, one can propose the following method of "conserving" the constant β in the segment of primary interest. We introduce the parameter $\beta' = \beta_1 A^\mu$ ($A = S/LH$ and μ is a real number). By a proper choice of μ, one can make a linear combination of β' and β_1 practically unchanged over the interval of heights of interest. Thus, in the interval $1.6R < H < 2R$, the quantities β_1 and $\beta_2 = \beta_1/A$ exhibit different behavior, but their linear combination $\beta_3 = (\beta_1 + \chi\beta_2)/(1 + \chi)$ for a properly chosen χ will be more conservative. For a cylindrical duct, we have $\chi = 1.65$ (see Table 1).

Thus, instead of (1.4), we use the following closure hypothesis:

$$\beta_3 = \beta_1 \frac{1+\chi/A}{1+\chi} = \frac{1+\chi/A}{1+\chi}\Big(-\frac{S}{\nu Q}\oint \nu \frac{\partial u}{\partial n}\,dl\Big) \qquad (\chi = 1.65). \tag{1.6}$$

Substituting (1.3), (1.5), and (1.6) into (1.2), we have the following system of dynamic equations:

$$\begin{gathered} \frac{\partial Q}{\partial t} + \gamma \frac{\partial}{\partial x}\Big(\frac{Q^2}{S}\Big) = Sg\Big(\sin\alpha - \frac{\partial H}{\partial x}\cos\alpha\Big) + \frac{\sigma}{\rho} S \frac{\partial^3 H}{\partial x^3} - \beta_3 \nu \frac{Q(1+\chi)}{S+\chi LH}, \\ \frac{\partial Q}{\partial x} + L\frac{\partial H}{\partial t} = 0. \end{gathered} \tag{1.7}$$

2. Dispersion Relation. Linearization of system (1.7) and substitution of it into a solution in the form $H = H_0 + H' \exp(ikx - i\omega t)$ lead to the dispersion relation

$$C^2 - \Big(2\gamma + \frac{\beta_1}{iK\mathrm{Re}}\Big)C + \gamma + \frac{\beta_1}{\mathrm{Re}}\Big(\frac{a}{iK} - A\cot\alpha - \frac{\mathrm{We}}{\sin\alpha}K^2\Big) = 0. \tag{2.1}$$

Here $A = S_0/(L_0H_0)$, $C = cS_0/Q_0 = \omega S_0/(kQ_0)$ and $K = kH_0$ are the dimensionless phase velocity and the wavenumber, and $\mathrm{Re} = Q_0/(\nu H_0)$ and $\mathrm{We} = \sigma S_0/(\rho g L_0 H_0^3)$ are the Reynolds and Weber numbers; the subscript 0 denotes the steady-flow parameters. The function $a(H)$ equals $1 + f_1$, where

$$f_1 = \frac{1 + \chi + 4\chi H_0(R - H_0)/L_0^2}{1 + \chi L_0 H_0/S_0}. \tag{2.2}$$

Equation (2.1) has two roots, which for small K assume the following approximate values:

$$C_1^{(+)} = 2\gamma - a + \frac{\beta_1}{iK\mathrm{Re}} + \frac{K}{i}\Big[\frac{\gamma - 2a\gamma + a^2}{(\beta_1/\mathrm{Re})} - A\cot\alpha\Big],$$

$$C_2^{(-)} = a - \frac{K}{i}\Big[\frac{\gamma - 2a\gamma + a^2}{(\beta_1/\mathrm{Re})} - A\cot\alpha\Big].$$

Since the amplitude of the disturbance is proportional to $\exp(ikx - i\omega t)$, the first solution always decays for small K since the exponent contains a negative term with the denominator containing the wavenumber

$$\exp(-i(2\gamma - a) - \beta_1/(K\mathrm{Re}) - K[(\gamma - 2a\gamma + a^2)/(\beta_1/\mathrm{Re}) - A\cot\alpha])t.$$

The second root gives the factor $\exp(-ai + K[(\gamma - 2a\gamma + a^2)/(\beta_1/\mathrm{Re}) - A\cot\alpha])t$.

Hence, the second solution can be unstable if the expression in square brackets is positive. What sign does this expression actually have?

Clearly, at small Reynolds numbers, the second term dominates, thus giving a negative sign to the whole expression. By contrast, at high Reynolds numbers, the sign of the second term or, more exactly, the sign of the function $\zeta(H) = \gamma - 2a\gamma + a^2$ is the determining factor.

Table 1 lists values of $f_1(H)$ and $\zeta(H)$. For $H > 1.6R$, we have $\zeta(H) > 0$, and $\zeta(H) \to +\infty$ as $H \to 2R$. This behavior of $\zeta(H)$ is explained by the fact that the nominator of expression (2.2) contains a negative (at $H > R$) term, whose absolute value increases infinitely as H approaches $2R$.

Thus, we can conclude that *in the model considered, for an arbitrary Reynolds number there is* $H_* >$ $1.6R$ *such that the flow becomes unstable against one-dimensional disturbances for liquid-layer heights* $H > H_*$.

3. Dispersion Curves. As follows from the aforesaid, the Reynolds number cannot serve as a stability criterion for the flows described by the model of one-dimensional waves. Stability against long-wave one-dimensional disturbances depends on two interrelated parameters: the Reynolds number and the dimensionless height of the liquid layer $\bar{H} = H/R$. Instability is observed at $\zeta(\bar{H})\,\mathrm{Re}(\bar{H}; R, g, \nu, \alpha) > \beta_1 A\cot\alpha$.

The Reynolds number can be expressed as the product of two quantities, Re_1 and Re_2, where Re_1 consists of a set of dimensionless parameters that depend only on $\bar{H}$, and Re_2 consists of a set of dimensional

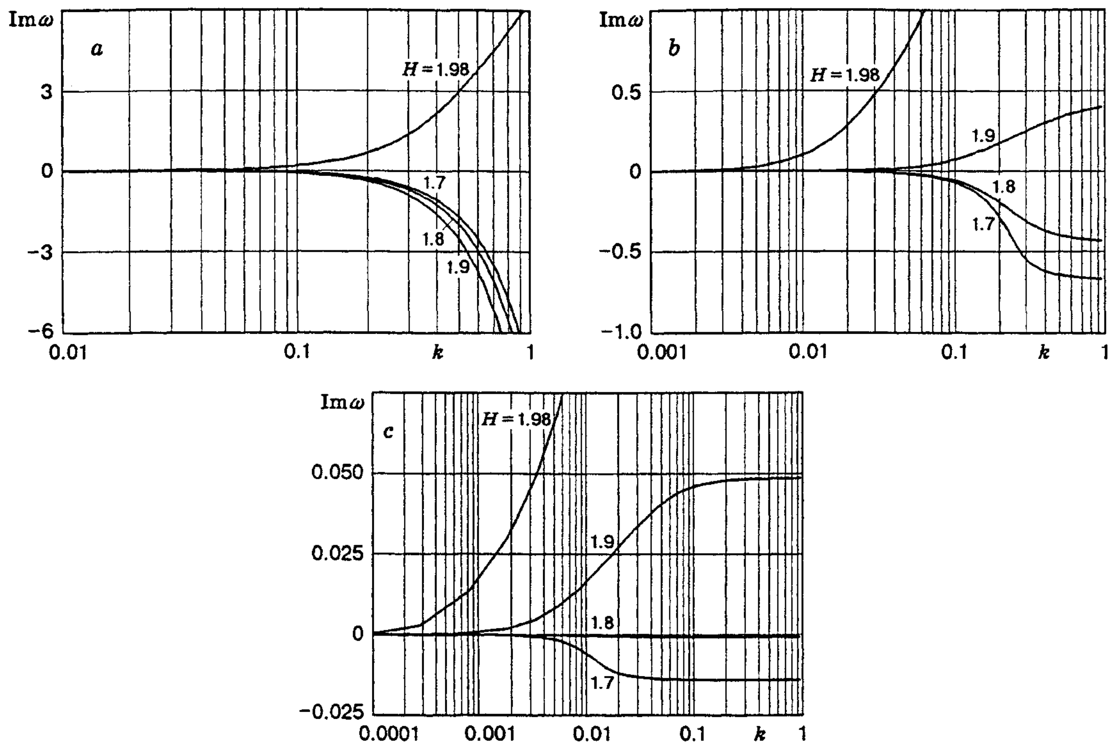

Fig. 2

parameters that characterizes the physical properties of the liquid (density and viscosity), the dimensions and position of the duct (R and α), and the external physical conditions (the presence of a gravity force):

$$\mathrm{Re} \doteq \mathrm{Re}_1(\bar{H})\,\mathrm{Re}_2(R, g, \nu, \alpha) = \frac{\bar{Q}_0}{\bar{H}_0}\,\frac{R^3 g \sin\alpha}{\nu^2}. \tag{3.1}$$

The Weber number can be represented in the same manner:

$$\mathrm{We} = \mathrm{We}_1(\bar{H})\,\mathrm{We}_2(R, g, \sigma, \rho) = \frac{\bar{S}_0}{\bar{L}_0 \bar{H}_0^3}\,\frac{\sigma}{\rho g R^2}.$$

Here $\bar{Q}_0 = Q_0/(v_* R^2)$, $v_* = R^2 g \sin\alpha/\nu$, $\bar{H}_0 = H_0/R$, $\bar{L}_0 = L_0/R$, and $\bar{S}_0 = S_0/R^2$.

The first term in (3.1) is called the *level* part of the Reynolds number, and the second the *metric* part. For a set of parameters g, ν, R, and α, the value of the Reynolds number for a duct half-filled with the liquid ($H = R$, $\bar{H} = 1$) is referred to as the *characteristic* Reynolds number Re_C. The same is true for the Weber number.

The characteristic Reynolds number depends on the physical and geometrical parameters g, ν, R, and α, i.e., on the metric part. In the numerical calculations, the metric parts of the Re and We numbers were such that the Re_C and We_C took the following values:

1) $\mathrm{Re}_C = 0.18$, $\mathrm{We}_C = 0.05$, $\alpha = 10°$;
2) $\mathrm{Re}_C = 5.74$, $\mathrm{We}_C = 0.005$, $\alpha = 10°$;
3) $\mathrm{Re}_C = 202.6$, $\mathrm{We}_C = 0.0001$, $\alpha = 1°$;
4) $\mathrm{Re}_C = 2016$, $\mathrm{We}_C = 0.0001$, $\alpha = 10°$.

These values correspond to the glycerin parameters ($\nu = 1.1 \cdot 10^{-3}$ m^2/sec, $\sigma = 59.4 \cdot 10^{-3}$ N/m, and $\rho = 1260$ kg/m^3).

Figure 2 shows the dependence of the imaginary part of the frequency $\mathrm{Im}\,\omega$ on the wavenumber k for various levels of the free surface and various characteristic Reynolds numbers: $\mathrm{Re}_C = 0.18$ (a), $\mathrm{Re}_C = 5.74$ (b), and $\mathrm{Re}_C = 202.6$ (c). Evidently, as Re_C increases, the critical liquid height that corresponds to neutral

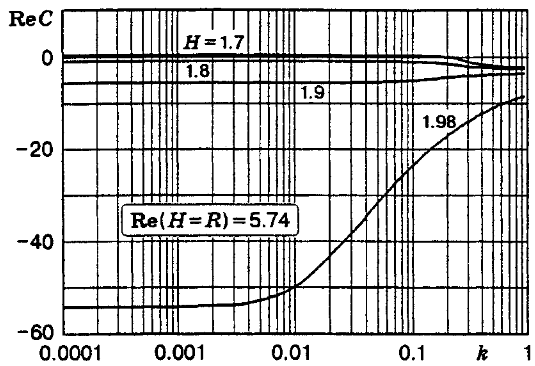

Fig. 3

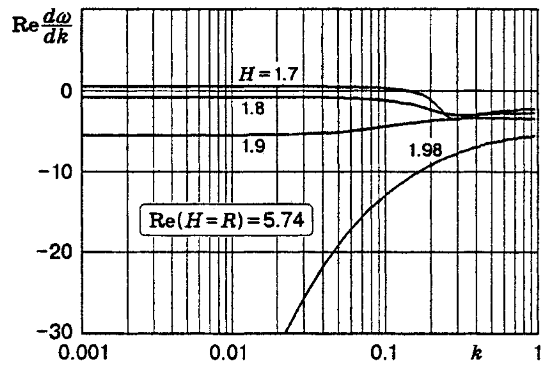

Fig. 4

disturbances decreases to a level $H = 1.6R$, i.e., to the point that corresponds to the maximum flow rate of the liquid. For very high Re_C, whose *level* part corresponds to $H < 1.6R$, all disturbances are nearly neutral.

For small Re_C and rather high free-surface level, the real parts of the phase ($\mathrm{Re}C$) (Fig. 3) and group [$\mathrm{Re}(d\omega/dk)$] (Fig. 4) velocities of propagation of disturbances are negative. Hence, such disturbances propagate upstream, which is consistent with the reasoning given in the introduction. This suggests that *at rather small Reynolds numbers, there is no steady regime of liquid flow in a fully filled duct with an open butt-end.* Accidental occurrence of an air interlayer at the duct outlet will inevitably cause lowering of the liquid level over the entire duct or transition to wave flow regimes. Obviously, at high Reynolds numbers, the disturbances are carried downstream, and, upstream, only very long waves can propagate when the height of the layer is close to the duct diameter.

The question of whether the above conclusions are applicable to a real situation, where the assumption of a horizontal free surface becomes false as H approaches $2R$, remains open. This issue is the subject of a separate study using a more complicated model.

This work was supported by the Russian Foundation for Fundamental Research (Grant No. 95-01-00879).

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